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On the isometric conjecture of Banach

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Let V be a Banach space all of whose subspaces of a fixed dimension n are isometric, with $1 < n < \dim(V)$. In 1932, S Banach asked if under this hypothesis V is necessarily a Hilbert space. In 1967, M Gromov answered it positively for even n. We give a positive answer for real V and odd n of the form $n = 4k + 1$, with the possible exception of $n = 133$. Our proof relies on a new characterization of ellipsoids in $\mathbb{R}^n$ for $n \geq 5$, as the only symmetric convex bodies all of whose linear hyperplane sections are linearly equivalent affine bodies of revolution.

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1 Introduction

S Banach asked in 1932 the following question:

Let V be a Banach space, real or complex, finite- or infinite-dimensional, all of whose n–dimensional subspaces, for some fixed integer n, $2 \leq n < \dim(V)$, are isometrically isomorphic to each other. Is it true that V is a Hilbert space? (See [3, page 244], or page 152 of the English translation, remarks on Chapter XII, property (5).)

It is important to note that Banach's question is a codimension 1 problem: since every Banach space all of whose subspaces of a fixed dimension $n \geq 2$ are Hilbert spaces is itself a Hilbert space,[1] an affirmative answer for n in codimension 1 implies immediately an affirmative answer for n in all codimensions.

The conjecture[2] was proved first for $n = 2$ and real V in 1935 by Auerbach, Mazur and Ulam [2] and for all $n \geq 2$ and infinite-dimensional real V in 1959 by A Dvoretzky [6; 7].

[1] This easily follows from the elementary characterization of a norm coming from an inner product via the "parallelogram law".

[2] Following a long-established tradition starting with [9], we rename Banach's question a "conjecture" in this article, although Banach himself, as far as we know, did not conjecture a positive answer.

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In 1967, M Gromov [9] proved the conjecture for even n and all V, real or complex, for odd n and real V with $\dim(V) \geq n+2$, and for odd n and complex V with $\dim(V) \geq 2n$ (which also proves the conjecture for all infinite-dimensional V, real or complex, as noted above). It is probably worth noticing that V Milman [16] extended Dvoretzky's theorem to the complex case, in particular reproving Banach's conjecture for infinite-dimensional complex V. A recent and very thorough account of the history of this conjecture is found in Soltan [22, Section 6, page 388]. We also recommend Pełciński [21] and the notes on Chapter 9 in Martini, Montejano and Oliveros [15, page 206].

Here we settle Banach's conjecture for real V and "one half" of the odd n, by showing that:

Main theorem *A real Banach space all of whose n–dimensional subspaces are isometrically isomorphic to each other for some fixed odd integer n of the form $n = 4k+1 \geq 5$ with $n \neq 133$ is a Hilbert space.*

Remark 1.1 The reason for the strange exception $n \neq 133$ will become clearer during the proof (133 is the dimension of the exceptional Lie group E_7).

Consider the closed unit ball $B = \{\|x\| \leq 1\} \subset V$. It is a symmetric convex body. Since a finite-dimensional Banach space is a Hilbert space if and only if B is an ellipsoid, Banach's question can be reformulated as:

Let $B \subset \mathbb{R}^N$ be a symmetric convex body all of whose sections by n–dimensional linear subspaces for some fixed integer n with $1 < n < N$ are linearly equivalent. Is it true that B is an ellipsoid?

Thus, in order to prove the main theorem, in the sequel we show the following:

Theorem 1.2 *Let $B \subset \mathbb{R}^{n+1}$ with $n = 4k+1 \geq 5$ and $n \neq 133$ be a convex symmetric body all of whose sections by n–dimensional subspaces are linearly equivalent. Then B is an ellipsoid.*

In fact, using Theorem 1 of Montejano [17], one can drop the symmetry assumption on B in the above reformulation, obtaining:

Main convex geometry theorem *Let $B \subset \mathbb{R}^{n+1}$ with $n = 4k+1 \geq 5$ and $n \neq 133$ be a convex body all of whose sections by n–dimensional affine subspaces through a fixed interior point are affinely equivalent. Then B is an ellipsoid.*

1.1 Sketch of the proof of the main theorem

Our proof of Theorem 1.2 combines two main ingredients: convex geometry and algebraic topology. To describe these, we need to recall first some standard definitions.

A *symmetric convex body* is a compact convex subset of a finite-dimensional real vector space with a nonempty interior, invariant under $x \mapsto -x$. A *hyperplane* is a codimension 1 linear subspace. An *affine hyperplane* is the translation of a hyperplane by some vector. A *hyperplane section* of a subset in a vector space is its intersection with a hyperplane. Two sets, each a subset of a vector space, are *linearly* (respectively, *affinely*) *equivalent* if they can be mapped to each other by a linear (respectively, affine) isomorphism between their ambient vector spaces. An *ellipsoid* is a subset of a vector space which is affinely equivalent to the unit ball in euclidean space.

A symmetric convex body $K \subset \mathbb{R}^n$ is a *symmetric body of revolution* if it admits an *axis of revolution*, ie a 1–dimensional linear subspace L such that each section of K by an affine hyperplane A orthogonal to L is an $(n-1)$–dimensional closed euclidean ball in A, centered at $A \cap L$ (possibly empty or just a point). If L is an axis of revolution of K then $L^\perp$ is the associated *hyperplane of revolution*. An *affine symmetric body of revolution* is a convex body linearly equivalent to a symmetric body of revolution. The images, under the linear equivalence, of an axis of revolution and its associated hyperplane of revolution of the body of revolution are an axis of revolution and associated hyperplane of revolution of the affine body of revolution (not necessarily perpendicular anymore). Clearly, an ellipsoid centered at the origin is an affine symmetric body of revolution and any hyperplane serves as a hyperplane of revolution.

With these definitions understood, the convex geometry result that we use in the proof of Theorem 1.2 is the following characterization of ellipsoids:

Theorem 1.3 *A symmetric convex body $B \subset \mathbb{R}^{n+1}$ with $n \geq 4$, all of whose hyperplane sections are linearly equivalent affine bodies of revolution, is an ellipsoid.*

The main ingredient in the proof of this theorem is the following result, possibly of independent interest:

Theorem 1.4 *Let $B \subset \mathbb{R}^{n+1}$ with $n \geq 4$ be a symmetric convex body, all of whose hyperplane sections are affine bodies of revolution. Then at least one of the sections is an ellipsoid.*

Note that in Theorem 1.4, unlike Theorem 1.3, we do not assume that all hyperplane sections of B are necessarily linearly equivalent to each other. If we add this assumption then it follows from Theorem 1.4 that *all* hyperplane sections of B are ellipsoids. It then follows easily that B *itself is an ellipsoid*: all hyperplane sections are Hilbert spaces and therefore V itself is also one.[3]

Theorem 1.4 is proved in Section 2. The rest of the article consists of topological methods to show that, under the hypotheses of Theorem 1.2, all hyperplane sections of B are necessarily affine symmetric bodies of revolution. The link to topology is via a beautiful idea that traces back to the work of Gromov [9]. It consists of the following key observation:

Lemma 1.5 *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body all of whose hyperplane sections are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. Let $G_K := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g(K) = K\}$ be the* ***group of linear symmetries*** *of K. Then the structure group of S^n can be reduced to G_K.*

See Section 3.1 below for a proof of this lemma, as well as a brief reminder about structure groups of differentiable manifolds and their reductions. Lemma 1.5 can be interpreted through the notion of *a field of convex bodies* tangent to S^n. See, for example, Mani [14] and Montejano [17].

Following Lemma 1.5, our task is to understand the possible reductions of the structure group of S^n (a classical problem in topology). The results we need are contained in the next purely topological theorem, which, when applied to Lemma 1.5 with the dimension hypothesis of Theorem 1.2, implies that K is an affine symmetric body of revolution.

But first another definition. We say that a subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ is *reducible* if the induced action on $\mathbb{R}^n$ leaves invariant a k–dimensional linear subspace for some $0 < k < n$; otherwise, it is an *irreducible* subgroup of $\mathrm{GL}_n(\mathbb{R})$. (Beware of the potentially confusing use of the notions "reducible" and "can be reduced" in the statement of the following theorem.)

[3] In fact, this classical result is known to hold (in every codimension) even without the symmetry assumption on B (see eg Theorem 2.12.4 of [15] or [22]). It is an open question whether a symmetric convex body all of whose sections are affine symmetric bodies of revolution is itself an affine body of revolution (the converse of Lemma 2.4). In Remark 2.9 we briefly discuss this question and explain why Theorem 1.4 may be considered as a first step towards an affirmative answer.

Theorem 1.6 *Let* $n \equiv 1 \bmod 4$ *with* $n \geq 5$*, and suppose that the structure group of* S^n *can be reduced to a closed connected subgroup* $G \subset \mathrm{SO}_n$*. Then:*

(a) *If* G *is reducible then it is conjugate to a subgroup of the standard inclusion* $\mathrm{SO}_{n-1} \subset \mathrm{SO}_n$*, acting transitively on* S^{n-2}*.*

(b) *If* G *is irreducible then* $G = \mathrm{SO}_n$*, or* $n = 133$ *and* $G \subset H \subset \mathrm{SO}_{133}$*, where* H *is the image of the adjoint representation of the simple exceptional Lie group* E_7*.*

We prove Theorem 1.6 in Section 3.2 by applying to our situation some known results from the literature about structure groups on spheres, mainly from Steenrod [23], Leonard [13] and Čadek and Crabb [5]. Furthermore, for case (b) (the irreducible case), we need to supplement these results with several facts about the representation theory and topology of compact Lie groups.

In summary, Theorem 1.2 is a consequence of the above results, as follows. Since all hyperplane sections of B are linearly equivalent to each other, they are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. By Lemma 1.5, the structure group of S^n can be reduced to G_K. It is easy to see that it can be further reduced to the identity component $G_K^0 \subset G_K$ (Remark 3.1). For a convex body K, G_K (and thus G_K^0) is compact (Lemma 2.1) and therefore G_K^0 is conjugate to a subgroup of SO_n (Lemma 2.2); hence, by passing to a convex body linearly equivalent to K, we can assume that $G_K^0 \subset \mathrm{SO}_n$. Next, Theorem 1.6 applied to $G = G_K^0$ implies that K is a symmetric body of revolution: in case (a), $\mathbb{R}e_n$ is an axis of revolution of K; in case (b), K is a euclidean ball. Thus all hyperplane sections of B are linearly equivalent to the symmetric body of revolution K. It follows, by Theorem 1.3, that B is an ellipsoid. □

Remark 1.7 As for the remaining cases, ie $n = 133$ or $n \equiv 3 \bmod 4$, we do not have much to say. In order to push our methods for the $n = 133$ case it would suffice to prove that S^{133} does not admit an E_7–structure. Perhaps a homotopy-theorist could settle this. To attack the cases with $n \equiv 3 \bmod 4$, completely different ideas should be used, since the topology of the tangent bundle of S^n is too trivial (eg S^3 and S^7 are parallelizable).

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2 Affine bodies of revolution

The aim of this section is to prove Theorem 1.3, announced in the introduction. For this purpose, we collect here the following lemmas.

2.1 Some preliminary lemmas

The first two lemmas are quite standard; we supply proofs for the convenience of the reader.

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body. Then its linear symmetry group $G_K = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g(K) = K\}$ is compact.*

Proof Let $A_K := \{a \in \mathrm{End}(\mathbb{R}^n) \mid a(K) \subset K\}$. Since K is closed in $\mathbb{R}^n$, A_K is closed in $\mathrm{End}(\mathbb{R}^n) \simeq \mathbb{R}^{n^2}$ (this follows easily from the continuity of matrix multiplication $\mathrm{End}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$). Since K is bounded and 0 is an interior point, there exist $R, r > 0$ such that $B_r \subset K \subset B_R$, where $B_\rho \subset \mathbb{R}^n$ is the closed ball of radius ρ centered at the origin. It follows that for every $a \in A_K$, $a(B_r) \subset B_R$, hence $\|a\| \le R/r$. Thus $A_K \subset \mathrm{End}(\mathbb{R}^n)$ is also bounded and hence compact. It remains to show that $G_K \subset A_K$ is closed. Let $g_i \in G_K$ with $g_i \to g \in \mathrm{End}(\mathbb{R}^n)$. Since $(g_i)^{-1} \in A_K$, $(g_i)^{-1}(B_r) \subset B_R$, hence $0 < (r/R)\|v\| \le \|g_i v\|$ for all i and all $v \ne 0$. Taking $i \to \infty$ we get $0 < (r/R)\|v\| \le \|gv\|$, hence g is invertible. Now, $g_i \to g$ implies $(g_i)^{-1} \to g^{-1}$ and thus $g^{-1} \in A_K$, ie $g^{-1}(K) \subset K$, which is equivalent to $K \subset g(K)$. Therefore $g \in G_K$. □

Lemma 2.2 *Every compact subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ is conjugate to a subgroup of O_n.*

Proof By taking an arbitrary positive inner product on $\mathbb{R}^n$ (eg the standard inner product $\sum x_i y_i$) and averaging it over G with respect to a bi-invariant measure, one obtains a G–invariant inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Now any two inner products on $\mathbb{R}^n$ are linearly isomorphic to each other, hence one can find an element $g \in \mathrm{GL}_n(\mathbb{R})$ such that $(u, v) \mapsto \langle gu, gv\rangle$ is the standard inner product on $\mathbb{R}^n$. It follows that $g^{-1}Gg \subset O_n$. For more details see eg Proposition 3.1 on page 36 of [1]. □

There is also an alternative geometric proof of Lemma 2.2 via the notion of minimal ellipsoids, as in [9, Lemma 1].

Lemma 2.3 *A symmetric affine body of revolution* $K \subset \mathbb{R}^n$ *with* $n \geq 3$, *admitting two different hyperplanes of revolution*, *is an ellipsoid.*

Proof Let $G_K = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g(K) = K\}$ and let $G = G_K^0$ be the identity component of G_K. By Lemma 2.2, G is conjugate to a subgroup of SO_n; we may assume, by passing to a body of revolution linearly equivalent to K, that $G \subset \mathrm{SO}_n$. We will show that in this case K is a ball centered at the origin, by showing that $G = \mathrm{SO}_n$.

Now, each hyperplane of revolution of K gives rise to a subgroup of G conjugate in SO_n to SO_{n-1} (the stabilizer of the hyperplane). Thus, our hypotheses imply that $\mathrm{SO}_{n-1} \subsetneq G \subset \mathrm{SO}_n$. But it is well known that SO_{n-1} is a *maximal connected* subgroup of SO_n, ie $G = \mathrm{SO}_n$ (see [19, Lemma 4, page 463]). □

Lemma 2.4 *Let* $K \subset \mathbb{R}^n$ *with* $n \geq 3$ *be an affine symmetric body of revolution. Then any section* $K' = \Gamma \cap K$ *with a* k*–dimensional linear subspace* $\Gamma \subset \mathbb{R}^n$, *for some* $1 < k < n$, *is an affine symmetric body of revolution in* Γ. *Furthermore*, *if* L *is an axis of revolution of* K *and* H *the associated hyperplane of revolution*, *then*:

(a) *If* $\Gamma \subset H$ *then* K' *is an ellipsoid.*

(b) *If* $\Gamma \not\subset H$ *then* $H' := \Gamma \cap H$ *is a hyperplane of revolution of* K'.

(c) *If* $L \subset \Gamma$ *then* L *is also the axis of revolution of* K' *associated to the hyperplane of revolution* $\Gamma \cap H$.

Proof (a) If $\Gamma \subset H$ then $\Gamma \cap K$ is a linear section of the ellipsoid $H \cap K$, hence is an ellipsoid.

(b) We can assume, by applying an appropriate linear transformation, that K is a symmetric body of revolution with an axis of revolution $L = \mathbb{R}e_n$ and plane of revolution $H = L^{\perp} = \{x_n = 0\}$ such that $H \cap K$ is the unit ball in H and $H \pm e_n$ are support hyperplanes of K at $\pm e_n$. Furthermore, we can also arrange that $H' := \Gamma \cap H$ is spanned by $e_1, \dots, e_{k-1}$ and so Γ is spanned by $e_1, \dots, e_{k-1}, v$, where $v = \lambda e_{n-1} + e_n$ for some $\lambda \in \mathbb{R}$. To show that H' is a hyperplane of revolution of K' with an associated axis of revolution $L' = \mathbb{R}v$, we need to show that every nonempty section of K' by an affine hyperplane of the form $H' + tv$ with $t \in \mathbb{R}$ is a $(k-1)$–dimensional ball in $H' + tv$, centered at tv. The latter section is the section of the $(n-1)$–dimensional ball $(H + te_n) \cap K$, centered at te_n, by $H' + tv$, a $(k-1)$–dimensional affine subspace of $H + te_n$, hence it is a $(k-1)$–dimensional ball, centered at tv, as needed.

(c) In the previous item, if $L \subset \Gamma$, we can choose $v = e_n$. □

Lemma 2.5 *Let $K \subset \mathbb{R}^n$ with $n \geq 3$ be an affine symmetric body of revolution with an axis of revolution L. Suppose a section of K by a linear subspace $\Gamma \subset \mathbb{R}^n$ of dimension ≥ 2 passing through L is an ellipsoid. Then K is an ellipsoid.*

Proof Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. By passing to a linearly equivalent body of revolution, we can assume that K is a symmetric body of revolution with an axis of revolution $L = \mathbb{R}e_n$ and associated hyperplane of revolution $H = L^{\perp} = \{x_n = 0\}$. Furthermore, we can also assume that $H \cap K$ is the unit ball in H and that $H \pm e_n$ are support hyperplanes of K at $\pm e_n$. We will show that, under these assumptions, K is the unit ball in $\mathbb{R}^n$. To this end, it is enough to show that each section of K by a 2–dimensional subspace Δ containing L is the unit disk in Δ centered at the origin. Let us choose a 2–dimensional subspace $\Delta \subset \Gamma$ containing L and a unit vector v in the 1–dimensional space $\Delta \cap H$. Then $\Delta \cap K$ is a (solid) ellipse, centered at the origin, whose boundary passes through $\pm v$ and $\pm e_n$, with support lines $\mathbb{R}v \pm e_n$ at $\pm e_n$. It follows that $\Delta \cap K$ is the unit disk in Δ centered at the origin. Now, since $L = \mathbb{R}e_n$ is an axis of revolution of K, all rotations in $\mathbb{R}^n$ about L leave K invariant. Applying all such rotations to Δ, we obtain all 2–dimensional subspaces containing L, and each of them intersects K in a unit disk centered at the origin, as needed. □

Lemma 2.6 *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body with $n \geq 4$ and $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{n+1}$ two distinct hyperplanes such that the hyperplane sections $K_i := \Gamma_i \cap B$ for $i = 1, 2$ are affine symmetric bodies of revolution with axes and associated hyperplanes of revolution L_i and H_i (respectively). If $L_1 \subset H_2$ then K_1 is an ellipsoid.*

Proof Let $E := K_1 \cap K_2$. We will show that E is an ellipsoid. This implies, by Lemma 2.5, that K_1 is an ellipsoid, since $E = K_1 \cap \Gamma_2$ and Γ_2 contains L_1, an axis of revolution of K_1.

To show that E is an ellipsoid, we note first that Γ_2 does not contain H_1, else $L_1, H_1 \subset \Gamma_2$ would imply $\Gamma_1 = L_1 \oplus H_1 \subset \Gamma_2$. Hence, by Lemma 2.4(b), $\Gamma_2 \cap H_1$ is a hyperplane of revolution of $E = \Gamma_2 \cap K_1$.

Next we look at $\Gamma_1 \cap \Gamma_2$. This has codimension 1 in Γ_2. If it coincides with H_2, then $E = \Gamma_1 \cap K_2 = H_2 \cap K_2$, which is an ellipsoid, by Lemma 2.4(a). If $\Gamma_1 \cap \Gamma_2 \neq H_2$, then, by Lemma 2.4(b), $\Gamma_1 \cap H_2$ is a hyperplane of revolution of $E = \Gamma_1 \cap K_2$.

Now $\Gamma_1 \cap H_2$ and $\Gamma_2 \cap H_1$ are two distinct hyperplanes of revolution of E, since L_1 is contained in the first but not in the second. It follows from Lemma 2.3 that E is an ellipsoid. □

The statement of the following lemma has appeared elsewhere (eg statement III of the proof of Theorem 2.2 of [18]), but we did not find a published proof of it (perhaps because it is intuitively clear and a hassle to prove).

Lemma 2.7 *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body and $x_i \to x$ a convergent sequence in S^n. Assume each hyperplane section $x_i^\perp \cap B$ is an affine symmetric body of revolution with an axis of revolution $L_i \subset x_i^\perp$. If $\{L_i\}$ is a convergent sequence in $\mathbb{R}P^n$, $L_i \to L$, then $x^\perp \cap B$ is an affine symmetric body of revolution with an axis of revolution L.*

Proof Let $\Gamma_i := x_i^\perp$, $\Gamma := x^\perp$, $K_i := \Gamma_i \cap B$ and $K := \Gamma \cap B$. Assume, without loss of generality, that $x = e_{n+1}$, so that $\Gamma = \mathbb{R}^n$.

Claim 1 *$K_i \to K$ in the Hausdorff metric.*

We postpone for the moment the proof of this claim and the two subsequent ones. Define $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Note that $\pi(K) = K$ and $\pi(L) = L$.

Claim 2 *For large enough i, $\pi|_{\Gamma_i} : \Gamma_i \to \mathbb{R}^n$ is a linear isomorphism.*

We henceforth restrict to a subsequence of $\{K_i\}$ such that each $\pi|_{\Gamma_i}$ is an isomorphism. Let $K_i' := \pi(K_i) \subset \mathbb{R}^n$ and $L_i' := \pi(L_i) \subset \mathbb{R}^n$. Then each $K_i' \subset \mathbb{R}^n$ is an affine symmetric body of revolution with an axis of revolution L_i', $L_i' \to L$ and $K_i' \to K$ (by Claim 1). By definition of affine symmetric body of revolution, there exist linear isomorphisms $T_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $K_i'' := T_i(K_i')$ is an (honest) symmetric body of revolution. By postcomposing T_i with appropriate elements of $\mathrm{GL}_n(\mathbb{R})$, we can also assume that $\mathbb{R}e_n = T_i(L_i')$ is an axis of revolution of K_i'', that $\mathbb{R}^{n-1} \pm e_n$ are support hyperplanes of K_i'' at $\pm e_n$ and that $K_i'' \cap \mathbb{R}^{n-1}$ is the unit $(n-1)$–dimensional closed ball in $\mathbb{R}^{n-1}$, centered at the origin.

Claim 3 *$\{T_i\}$ is contained in a compact subset of $\mathrm{GL}_n(\mathbb{R})$.*

It follows that there is a subsequence of $\{T_i\}$, which we rename $\{T_i\}$, converging to an element $T \in \mathrm{GL}_n(\mathbb{R})$. Let $K'' := T(K)$. Then $\lim K_i'' = \lim T_i(K_i') = (\lim T_i)(\lim K_i') = T(K) = K''$, and $T(L) = (\lim T_i)(\lim L_i') = \lim T_i(L_i) = \mathbb{R}e_n$. It is thus enough to show that $\mathbb{R}e_n$ is an axis of revolution of K''. Now $\mathbb{R}e_n$ is an axis of revolution of each K_i'', hence $gK_i'' = K_i''$ for all $g \in O_{n-1}$ (the elements of O_n leaving $\mathbb{R}e_n$ fixed). Taking the limit $i \to \infty$ we obtain $g(K'') = K''$. Hence $\mathbb{R}e_n$ is an axis of revolution of K''.

Proof of Claim 1 We first show that $d(y,\Gamma_i)\to 0$ for every $y\in\Gamma$. Let $y_i\in\Gamma_i$ be the orthogonal projection of y onto Γ_i. Then $y=y_i+d(y,y_i)x_i$, hence $0=\langle y_i,x_i\rangle=\langle y-d(y,y_i)x_i,x_i\rangle=\langle y,x_i\rangle-d(y,y_i)$. It follows that $d(y,\Gamma_i)\le d(y,y_i)=\langle y,x_i\rangle\to\langle y,x\rangle=0$.

Now assume K_i does not converge to K. There is then an $\epsilon>0$ and a subsequence of the K_i, renamed K_i, such that $d_H(K_i,K)\ge\epsilon$ for all i, where d_H is the Hausdorff metric. By definition of d_H, there is then for each i either

(a) $k_i\in K$ such that $d(k_i,K_i)\ge\epsilon$, or

(b) $k_i'\in K_i$ such that $d(k_i',K)\ge\epsilon$.

At least one of these two cases must occur infinitely often.

If (a) occurs infinitely often, then, by compactness of K, there is a subsequence of the k_i, renamed k_i, such that $k_i\to k\in K$ and $d(k_i,k)<\frac{1}{2}\epsilon$ for all i. It follows that $B_{\epsilon/2}(k)\subset B_\epsilon(k_i)$ for all i, hence $B_{\epsilon/2}(k)\cap K_i=\varnothing$ for all i. Clearly, $k\neq 0$. Consider the line segment $[0,k]\subset K$. Then $[0,k)\subset\operatorname{int}(B)$ (this holds for every point $k\neq 0$ of a symmetric convex body B). Let $k'\in[0,k)\cap B_{\epsilon/2}(k)$. Then $k'\in\operatorname{int}(B)\cap B_{\epsilon/2}(k)$, hence there exists $\delta>0$ such that $B_\delta(k')\subset B\cap B_{\epsilon/2}(k)$. It follows that $B_\delta(k')\cap\Gamma_i=(B_\delta(k')\cap B)\cap\Gamma_i=B_\delta(k')\cap(B\cap\Gamma_i)=B_\delta(k')\cap K_i\subset B_{\epsilon/2}(k)\cap K_i=\varnothing$ for all i. This contradicts $d(k',\Gamma_i)\to 0$ of the first paragraph above.

If case (b) occurs, we have a sequence $k_i'\in K_i\subset B$ with $d(k_i',K)\ge\epsilon$. From compactness of B, and passing to a subsequence, we may assume that $k_i'\to b\in B$, with $d(b,K)\ge\epsilon$ holding. Moreover, since $0=\langle k_i',x_i\rangle\to\langle b,x\rangle$, we see that $b\in\Gamma\cap B=K$. But this is a contradiction. ◁

Proof of Claim 2 $\operatorname{Ker}(\pi)=\mathbb{R}e_{n+1}$, hence $\operatorname{Ker}(\pi|_{\Gamma_i})\neq 0$ if and only if $e_{n+1}\perp x_i$. But $x_i\to e_{n+1}$ implies $\langle x_i,e_{n+1}\rangle\to 1$, hence $\langle x_i,e_{n+1}\rangle\neq 0$ for all i sufficiently large. ◁

Proof of Claim 3 For each pair of constants $c,C>0$, the set of elements $A\in\mathrm{GL}_n(\mathbb{R})$ satisfying $c\|v\|\le\|Av\|\le C\|v\|$ for all $v\in\mathbb{R}^n$ is clearly closed. It is also bounded because its elements satisfy $\|A\|\le C$ (using the operator norm on $\operatorname{End}(\mathbb{R}^n)$). It is thus enough to find constants $c,C>0$ such that $c\|v\|\le\|T_iv\|\le C\|v\|$ for all $v\in\mathbb{R}^n$ and all i.

Denote by B_ρ the closed ball in $\mathbb{R}^n$ of radius ρ centered at the origin. Then there are constants $r',R',r'',R''>0$ such that $B_{r'}\subset\pi(B)\subset B_{R'}$ and $B_{r''}\subset K_i''\subset B_{R''}$

for all i. It follows that $T_i(B_{r'}) \subset T_i(K_i') = K_i'' \subset B_{R''}$, thus $\|T_i v\| \le C\|v\|$ for all $v \in \mathbb{R}^n$ and all i, where $C = R''/r'$.

Next, $(T_i)^{-1} B_{r''} \subset (T_i)^{-1}(K_i'') = K_i' \subset B_{R'}$, hence $\|(T_i)^{-1} w\| \le c'\|w\|$ for all $w \in \mathbb{R}^n$ and all i, where $c' = R'/r''$. Substituting $w = T_i v$ in the last inequality, we obtain $c\|v\| \le \|T_i v\|$ for all $v \in \mathbb{R}^n$ and all i, where $c = 1/c' = r''/R'$. □

Lemma 2.8 *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body all of whose hyperplane sections are nonellipsoidal affine symmetric bodies of revolution. For each $x \in S^n$ let L_x be the (unique) axis of revolution of $x^\perp \cap B$. Then $x \mapsto L_x$ is a continuous function $S^n \to \mathbb{R}P^n$.*

Proof Let $x_i \to x$ be a converging sequence in S^n. To show that $L_{x_i} \to L_x$, it is enough to show that L_{x_i} is convergent and its limit is an axis of revolution of $x^\perp \cap B$. Since $\mathbb{R}P^n$ is a compact metric space, to show that L_{x_i} is convergent it is enough to show that all its convergent subsequences have the same limit. To show this, it is enough to show that the limit of a convergent subsequence of L_{x_i} is an axis of revolution of $x^\perp \cap B$. This is the statement of Lemma 2.7. □

2.2 The proof of Theorem 1.3

We first show Theorem 1.4, ie assume $B \subset \mathbb{R}^{n+1}$ is a symmetric convex body all of whose hyperplane sections are affine symmetric bodies of revolution, and show that at least one of the hyperplane sections is an ellipsoid. If none of the sections is an ellipsoid, then, by Lemma 2.3, for each $x \in S^n$ the section $x^\perp \cap B$ has a unique axis of revolution $L_x \subset x^\perp$. By Lemma 2.8, $x \mapsto L_x$ defines a continuous function $S^n \to \mathbb{R}P^n$, ie a line subbundle of TS^n. (Note that for even n this is already a contradiction, so we proceed for odd n.) Now every line bundle on S^n with $n \ge 2$ is trivial, ie admits a nonvanishing section, hence one can find a continuous function $\psi\colon S^n \to S^n$ such that $\psi(x) \in L_x$ for all $x \in S^n$. Since $\psi(x) \perp x$, the function $F(t,x) := (t\psi(x) + (1-t)x)/\|t\psi(x) + (1-t)x\|$ for $0 \le t \le 1$ is well defined (the denominator does not vanish), defining a homotopy between $\psi = F(1,\cdot)$ and the identity map $F(0,\cdot)$. It follows that ψ is a degree 1 map and is thus *surjective*.

Now let $\Gamma_2 \cap B$ be a hyperplane section of B, with hyperplane of revolution $H_2 \subset \Gamma_2$. Let $L_1 \subset H_2$ be any 1–dimensional subspace. Then the surjectivity of ψ implies that B admits a hyperplane section $K_1 = \Gamma_1 \cap B$ with axis of revolution L_1. By Lemma 2.6, K_1 is an ellipsoid, in contradiction to our assumption that none of the hyperplane sections of B is an ellipsoid. This completes the proof of Theorem 1.4.

To complete the proof of Theorem 1.3, we use Theorem 1.4 to conclude that all hyperplane sections of B are ellipsoids, and hence that B itself is an ellipsoid, as needed. □

Remark 2.9 Lemma 2.4 says that any hyperplane section of an affine symmetric convex body of revolution B is again an affine symmetric convex body of revolution. The converse of this result, as far as we know, is an open problem. Let us state a somewhat more general question:

Let $B \subset \mathbb{R}^{n+1}$ with $n \geq 4$ be a convex body containing the origin in its interior. If every hyperplane section of B is an affine body of revolution, is B necessarily an affine body of revolution?

An obvious necessary condition for B to be an affine body of revolution is that one of its hyperplane sections is an ellipsoid (take the hyperplane of revolution of B). Thus, Theorem 1.4 can be viewed as a first step for a positive answer to the above question (at least, under the further assumption of symmetry). Since Theorem 1.4 assumes $n \geq 4$, we dare only ask the above question under the same dimension restriction.

The case $n = 2$ has a different flavor altogether, where "axis of revolution" of a plane section is replaced by "axis of symmetry". (For example, there are convex plane regions with several different axes of symmetry which are not ellipses; this is the reason we proved Theorem 1.4 only for $n \geq 4$.) Yet there is a result in this dimension, somewhat related to Theorem 1.4. It is Theorem 2.1 of [18]: *Let $B \subset \mathbb{R}^3$ be a convex body such that every plane section through some fixed interior point of B has an axis of symmetry. Then at least one of the sections is a disk.*

3 Structure groups of spheres

3.1 A reminder on structure groups of manifolds and their reduction

First, let us recall the following basic definitions (see, for example, Section 5 of Chapter I of [12] or Part I of [23]).

Let G be a topological group, M a topological space and $P \to M$ a principal G–bundle. A *reduction of the structure group* of $P \to M$ to a closed subgroup $H \subset G$ is a principal H–subbundle of P. Equivalently, it is a continuous section of the bundle $P/H \to M$ associated with the left G–action on G/H. The *frame bundle* of an n–dimensional differentiable manifold M is the $\mathrm{GL}_n(\mathbb{R})$–principal bundle $F(M) \to M$, whose fiber at a point $x \in M$ is the set of all linear isomorphisms $\mathbb{R}^n \to T_x M$, with the right $\mathrm{GL}_n(\mathbb{R})$–action given by precomposition of linear maps. A *G–reduction* of the structure group of a smooth n–manifold M (or a *G–structure*) is the reduction of the structure group $\mathrm{GL}_n(\mathbb{R})$ of its frame bundle to a closed subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$. Equivalently, it is given by an open cover of M, together with a trivialization of the restriction of TM to each of the covering open subsets, such that the transition functions between the trivializations on overlapping members of the cover take values in G [12, Proposition 5.3, page 53].

Remark 3.1 For $M = S^n$, there is a standard cover by two "hemispheres", intersecting along a neighborhood of the "equator" S^{n-1}, hence its structure group is given by a single transition function $\chi_n \colon S^{n-1} \to \mathrm{GL}_n(\mathbb{R})$, called the *characteristic map* [23, Section 18, pages 96–100]. The structure group of S^n can be reduced to G if and only if the characteristic map χ_n is homotopic to a map whose image is contained in G. In particular, for $n \geq 2$, since S^{n-1} is connected, if the structure group of S^n can be reduced to some closed subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ then it can be further reduced to its identity component $G^0 \subset G$.

Let us recall Lemma 1.5, announced in the introduction. It follows from Lemma 2 of [9], but since it is such a key result in this article, we offer here an alternative proof, somewhat more elementary and detailed.

Lemma 1.5 *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body all of whose hyperplane sections are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. Let $G_K := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g(K) = K\}$ be the group of linear symmetries of K. Then the structure group of S^n can be reduced to G_K.*

Proof Identify for each $x \in S^n$, by parallel translation in $\mathbb{R}^{n+1}$, the tangent space to S^n at x with $x^\perp \subset \mathbb{R}^{n+1}$ and define the set $P_x \subset F_x(S^n)$ of frames at x as the set of linear isomorphisms $\mathbb{R}^n \to x^\perp$ mapping K to $x^\perp \cap B$. Note that if $\phi \in P_x$ then $P_x = \phi\, G_K$, that is, $\sigma \colon x \mapsto P_x$ is a section of $F/G_K \to S^n$. In order to show that $P = \bigcup P_x \subset F$ is a G_K–reduction it is thus enough to show that

(1) G_K is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, and
(2) $\sigma \colon S^n \to F/G_K$ is continuous

(see eg [10, Theorem 2.3, page 74] or [23, Corollary 9.5, page 43]). By Lemma 2.1 above, G_K is a compact group, hence it is closed in $\mathrm{GL}_n(\mathbb{R})$.

To prove the continuity of σ, it is enough to show that for every convergent sequence $x_i \to x$ in S^n there exists a subsequence of $\{\sigma(x_i)\}$ converging to $\sigma(x)$. Let $\pi\colon F(S^n) \to F(S^n)/G_K$ be the natural projection and choose arbitrary lifts $\phi_i \in P_{x_i}$ of $\sigma(x_i)$. By the continuity of π, it is enough to find a subsequence of $\{\phi_i\}$ converging to an element $\phi \in P_x$.

Now each ϕ_i is a linear isomorphism $\mathbb{R}^n \to x_i^\perp \subset \mathbb{R}^{n+1}$, thus we may think of $\phi_i \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $\phi_i(K) \subset B$, with $\mathrm{int}(K) \neq \varnothing$ and B compact, and hence bounded, $\{\phi_i\}$ is a bounded set in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (K contains some basis β of $\mathbb{R}^n$ and $\phi_i(\beta) \subset B$). Therefore, $\{\phi_i\}$ has a convergent subsequence which we denote by ϕ_i as well, $\phi_i \to \phi$, for some $\phi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n+1})$. It remains to show that $\phi \in P_x$, ie ϕ is a linear isomorphism $\mathbb{R}^n \to x^\perp$ such that $\phi(K) = x^\perp \cap B$.

Let $K_i = x_i^\perp \cap B$, $K_\infty = x^\perp \cap B$. In the proof of Lemma 2.7 (Claim 1) we showed that $x_i \to x$ implies $K_i \to K_\infty$ (in the Hausdorff metric). Thus, $\phi(K) = (\lim \phi_i)(K) = \lim(\phi_i(K)) = \lim K_i = K_\infty$. Since K_∞ has nonempty interior in $x^\perp$, $\phi(K) = K_\infty$ implies that ϕ is a linear isomorphism $\mathbb{R}^n \to x^\perp$. Thus $\phi \in P_x$, as needed. □

3.2 Proof of Theorem 1.6(a) (the reducible case)

Suppose the structure group of S^n can be reduced to a closed connected subgroup $G \subset \mathrm{SO}_n$, acting reducibly on $\mathbb{R}^n$. Then G is conjugate to a closed connected subgroup $G' \subset \mathrm{SO}_k \times \mathrm{SO}'_{n-k} \subset \mathrm{SO}_n$ for some k with $\frac{1}{2}n \le k < n$, where SO'_{n-k} denotes the subgroup of SO_n fixing $\mathbb{R}^k = \{x_{k+1} = \cdots = x_n = 0\} \subset \mathbb{R}^n$. If $n \equiv 1 \bmod 4$, then such a reduction is possible only if $k = n-1$, ie $G' \subset \mathrm{SO}_{n-1}$, acting irreducibly on $\mathbb{R}^{n-1}$ (see [23, Sections 27.14 and 27.18, pages 143–144]). In particular, the structure group of S^n reduces to SO_{n-1} but not to SO_{n-2}. We shall next prove that G' acts transitively on S^{n-2} (see Corollary 3.2 of [13]).

Consider the standard fibration $\mathrm{SO}_{n-2} \to \mathrm{SO}_{n-1} \xrightarrow{\pi} S^{n-2}$. If G' does not act transitively on S^{n-2}, it means that the composition $G' \xhookrightarrow{i} \mathrm{SO}_{n-1} \xrightarrow{\pi} S^{n-2}$ is not surjective, and is therefore null-homotopic. Let $F\colon G' \times I \to S^{n-2}$ be the homotopy. Then, by the homotopy lifting property, there exists a map $\widetilde{F}$ completing the diagram

$$\begin{array}{ccc} G' & \xrightarrow{\;i\;} & \mathrm{SO}_{n-1} \\ {\scriptstyle I\times 0}\downarrow & \overset{\widetilde{F}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ G' \times I & \xrightarrow[F]{} & S^{n-2} \end{array}$$

Commutativity of the diagram implies that $\widetilde{F}(x,1) \in \mathrm{SO}_{n-2} \subset \mathrm{SO}_{n-1}$ for every $x \in G'$. Let $f : G' \to \mathrm{SO}_{n-2}$ be defined by $f(x) = \widetilde{F}(x,1)$; then, up to homotopy, the following diagram commutes:

$$\begin{array}{ccc} G' & \xrightarrow{\quad i \quad} & \mathrm{SO}_{n-1} \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle j} & \\ & \mathrm{SO}_{n-2} & \end{array}$$

But now precomposing $j \circ f$ with the characteristic map $\chi_n : S^{n-1} \to G'$ yields a reduction of the structure group of S^n to SO_{n-2},

$$\begin{array}{cccc} S^{n-1} \xrightarrow{\chi_n} & G' & \xrightarrow{\quad i \quad} & \mathrm{SO}_{n-1} \\ & & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle j} & \\ & & \mathrm{SO}_{n-2} & \end{array}$$

which is a contradiction. □

3.3 Proof of Theorem 1.6(b) (the irreducible case)

We start with the following three preliminary lemmas.

Lemma 3.2 *For all* $n \equiv 1 \bmod 4$ *with* $n \geq 5$, *if the structure group of* S^n *can be reduced to* $G \subset \mathrm{SO}_n$, *then* $\dim G \geq n-2$.

Proof This follows readily from Proposition 3.1 of [5], since — as mentioned above — the structure group of S^n for $n \equiv 1 \bmod 4$ may be reduced to SO_{n-1} but not to SO_{n-2}. Given that the argument is a simple one, we include it here.

Assume that $\dim G = k < n$. We are going to show that the structure group of S^n reduces to the standard $\mathrm{SO}_{k+1} \subset \mathrm{SO}_n$. This implies the result.

Consider the characteristic map $\chi_n : S^{n-1} \to \mathrm{SO}_n$ of S^n. Assuming that the structure group of S^n reduces to G amounts to assuming the existence of $f : S^{n-1} \to G$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad \chi_n \quad} & \mathrm{SO}_n \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & G & \end{array}$$

The standard inclusion $\mathrm{SO}_{k+1} \hookrightarrow \mathrm{SO}_n$ induces isomorphisms $\pi_j(\mathrm{SO}_{k+1}) \simeq \pi_j(\mathrm{SO}_n)$ for every $j < k$ (this follows immediately from the long exact sequences of the fibrations $\mathrm{SO}_{k+1+r} \to \mathrm{SO}_{k+2+r} \to S^{k+1+r}$ for the range of j's in question).

Now, this implies that $G \hookrightarrow \mathrm{SO}_n$ factors (up to homotopy) through SO_{k+1}. One way of seeing this is via obstruction theory. Think of G as a CW–complex. Then the obstruction to extend the inclusion $G \hookrightarrow \mathrm{SO}_{k+1}$ from the j–skeleton to the $(j+1)$–skeleton is a cocycle with coefficients in $\pi_j(\mathrm{SO}_{k+1})$. But the inclusion $\mathrm{SO}_{k+1} \hookrightarrow \mathrm{SO}_n$ induces isomorphisms onto $\pi_j(\mathrm{SO}_n)$ for $j < k$, where we know that the obstruction vanishes. Therefore, there is no obstruction to construct $G \to \mathrm{SO}_{k+1}$ such that $G \to \mathrm{SO}_{k+1} \hookrightarrow \mathrm{SO}_n$ is homotopic to the inclusion $G \hookrightarrow \mathrm{SO}_n$. Hence, the structure group of S^n reduces to SO_{k+1}. □

Lemma 3.3 *If $n \geq 8$, then the structure group of S^n cannot be reduced to an irreducible subgroup $G \subsetneq \mathrm{SO}_n$ isomorphic to SO_k, SU_m or Sp_m, with $k \geq 4$ and $m \geq 2$.*

Proof This is Corollary 2.2 of [5]. □

Lemma 3.4 *For all $n \geq 2$, if the structure group of S^n reduces to a closed connected irreducible **maximal** subgroup $H \subsetneq \mathrm{SO}_n$, then H is **simple**.*

Proof See Theorem 3 of [13]. □

We now proceed to the proof of Theorem 1.6(b), using the above three lemmas. We first treat $n \geq 9$, then $n = 5$.

The case $n \geq 9$ Assume that $G \subset \mathrm{SO}_n$ acts irreducibly on $\mathbb{R}^n$ but is not all of SO_n. Then it is contained in some *maximal* connected closed subgroup H, $G \subset H \subsetneq \mathrm{SO}_n$. The structure group of S^n then reduces to H, acting also irreducibly on $\mathbb{R}^n$. By Lemma 3.4, H is simple. By Lemma 3.3, H is a nonclassical group, ie it is isomorphic to either Spin_m with $m \geq 7$, or one of the five exceptional simple Lie groups, G_2, F_4, E_6, E_7 or E_8. By Lemma 3.2, $n \leq \dim H + 2$. Let V be the complexification of the (irreducible) representation of H on $\mathbb{R}^n$. Since $\dim V$ is odd, V is a complex irreducible representation.

Let us list all the properties of the pair (H, V) that we have so far:

(i) H is a nonclassical compact connected group, ie Spin_m with $m \geq 7$, or one of the five exceptional compact simple Lie groups.

(ii) V is a complex irreducible representation of H of *real type* (ie the complexification of a real irreducible representation).

(iii) $\dim V \equiv 1 \mod 4$.

(iv) $\dim V \leq \dim(H) + 2$.

(v) If $H = \mathrm{Spin}_m$, then its action on V does not factor through SO_m.

We claim that these five conditions on the pair (H, V) are *incompatible* for $\dim V \geq 9$, except if V is the complexified adjoint representation of $H = E_7$, in which case $\dim V = \dim H = 133 \equiv 1 \mod 4$. We are unable to exclude this case.

For the exceptional groups, one can simply check (eg in Wikipedia) that none of them, other than E_7, has a nontrivial irreducible representation satisfying conditions (iii) and (iv). In the following table we list the smallest irreducible representations for them; we have marked in boldface the first dimensions that are $\equiv 1 \mod 4$:

group	G_2	F_4	E_6	E_7	E_8
$\dim H$	14	52	78	133	248
Irreps	7	26	27	56	248
	14	52	78	**133**	3875
	27	**273**	351	912	⋮
	64	⋮	**2925**	⋮	**1 763 125**
	77	⋮	⋮	⋮	⋮

For the spin groups, the next lemma shows that conditions (iii) and (v) are incompatible. (We thank Ilia Smilga for kindly informing us about this lemma and its proof.)

Lemma 3.5 *Every irreducible complex representation of* Spin_m *with* $m \geq 3$ *which does not factor through* SO_m *is even-dimensional.*

Proof We first review some well-known general facts concerning representations of simple compact Lie groups (see for example [1]). To each d–dimensional complex representation of a compact semisimple Lie group G of rank r with a maximal torus T one can associate its weight system $\Omega \subset \mathfrak{t}^*$, a subset with d points (counting multiplicity). The Weyl group $W = N_G(T)/T$ acts on $\mathfrak{t}^*$, preserving Ω. Thus, to show that d is even, it is enough to show the following:

(a) An irreducible nonclassical representation V of Spin_m does not have a 0 weight.

(b) The Weyl group of Spin_m contains a subgroup whose order is a positive power of 2, and whose only fixed point in $\mathfrak{t}^*$ is 0.

Note that (a) and (b) imply that d is even, since under the action of said subgroup of W, say W', Ω breaks into the disjoint union of W'–orbits, each with an even number of elements, since, by (a), all stabilizers are strict subgroups of W', hence have even index.

To show (a), note that the T–action on the 0 weight space is trivial. Now $-1 \in \mathrm{Spin}_m$ is in T (since it is central), but -1 must act on V by $-\mathrm{Id}$, else the Spin_m–action on V would factor through $\mathrm{SO}_m = \mathrm{Spin}_m/\{\pm 1\}$.

To show (b), let us first take $m = 2k$. Then $\mathbb{R}^m$ decomposes under T as the direct sum of k 2–planes. Consider the subgroup $N' \subset \mathrm{SO}_m$ which leaves invariant each of these 2–planes. Then $N' \simeq S(O_2 \times \cdots \times O_2)$, $T \subset N' \subset N(T)$, and its image $W' = N'/T \subset W = N(T)/T$ acts on $\mathfrak{t}^*$ by diagonal matrices with entries ± 1 on the diagonal, with an even number of -1's. Using this description, it is easy to show that W' has order 2^{k-1} and that its only fixed point in $\mathfrak{t}^*$ is 0.

For $m = 2k+1$ the argument is simpler. Under T, $\mathbb{R}^m$ decomposes as a direct sum of k 2–planes, plus a line. We take an element in SO_m which is a reflection about a line through the origin in each of these planes, and $(-1)^k$ in the line. This is in $N(T)$ and acts on $\mathfrak{t}^*$ by $-\mathrm{Id}$, hence its image in W has order 2 and its only fixed point in $\mathfrak{t}^*$ is the origin. □

The case $n = 5$ The only reduction of the structure group of S^5 that cannot be ruled out by Lemmas 3.2, 3.3 or 3.4 is the 5–dimensional irreducible representation of SO_3. This case is eliminated by the next lemma.

Lemma 3.6 *Let $\rho\colon \mathrm{SO}_3 \to \mathrm{SO}_5$ be the irreducible 5–dimensional representation of SO_3. Then, for any $f\colon S^4 \to \mathrm{SO}_3$, the composition $S^4 \xrightarrow{f} \mathrm{SO}_3 \xrightarrow{\rho} \mathrm{SO}_5$ is null-homotopic. It follows that the structure group of S^5 cannot be reduced to ρ.*

Proof Since the tangent bundle of S^5 is not trivial, the characteristic map $\chi_5\colon S^4 \to \mathrm{SO}_5$ is not null-homotopic. Consequently, to show that the structure group of S^5 cannot be reduced to ρ it is enough to show that any composition $S^4 \xrightarrow{f} \mathrm{SO}_3 \xrightarrow{\rho} \mathrm{SO}_5$ is null-homotopic. To show this, we use the following three claims:

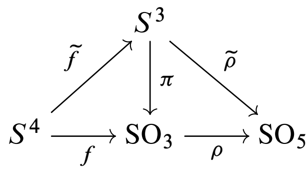
(a) $\pi_3(S^3) \simeq \pi_3(\mathrm{SO}_3) \simeq \pi_3(\mathrm{SO}_5) \simeq \mathbb{Z}$ and $\pi_4(S^3) \simeq \pi_4(\mathrm{SO}_3) \simeq \pi_4(\mathrm{SO}_5) \simeq \mathbb{Z}_2$.

(b) The map $\rho_*\colon \pi_3(\mathrm{SO}_3) \to \pi_3(\mathrm{SO}_5)$ has a cyclic cokernel of *even* order (the "Dynkin index" of ρ).

(c) For any topological group G and integers $k, n \geq 2$, the composition of maps $S^n \to S^k \to G$ defines a *biadditive* map $\pi_k(G) \times \pi_n(S^k) \to \pi_n(G)$, $([f],[g]) \mapsto [f] \circ [g] := [f \circ g]$ (the "composition product").

Claim (a) is standard (see eg [11, Volume 2, Appendix A, Table 6.VII, page 1745]). Claim (b) is a straightforward Lie algebraic calculation, see next subsection. For claim (c), see [24, Theorem (8.3), page 479].

Now let $f\colon S^4 \to \mathrm{SO}_3$ be any (pointed) continuous map and $\tilde{f}\colon S^4 \to S^3$ its lift to the universal double cover $\pi\colon S^3 \to \mathrm{SO}_3$. By (b), the composition $\tilde{\rho} := \rho \circ \pi\colon S^3 \to \mathrm{SO}_5$ as in



has an even Dynkin index (in fact, it is the same as the index of ρ, since π, being a cover, has index 1). In particular, $[\tilde{\rho}] = 2[u] \in \pi_3(\mathrm{SO}_5)$ for some $u\colon S^3 \to \mathrm{SO}_5$. By (c), with $n = 4$, $k = 3$ and $G = \mathrm{SO}_5$, $[\rho \circ f] = [\tilde{\rho} \circ \tilde{f}] = [\tilde{\rho}] \circ [\tilde{f}] = (2[u]) \circ [\tilde{f}] = 2([u] \circ [\tilde{f}]) = 0 \in \pi_4(\mathrm{SO}_5) \simeq \mathbb{Z}_2$. □

A byproduct of the proof of Theorem 1.6 is the following corollary, which could be of some interest to topologists:

Corollary 3.7 *Suppose that the structure group of S^n can be reduced to a closed connected subgroup $G \subsetneq \mathrm{SO}_n$. If $n = 4k+1 \geq 5$, but $n \neq 9, 17$ or 133, then G is conjugate to the standard inclusion of SO_{4k}, U_{2k} or SU_{2k} in SO_{4k+1}. For $n = 9$, G is conjugate to the standard inclusion of SO_8, U_4, SU_4 or $\mathrm{Spin}_7 \subset \mathrm{SO}_8$ in SO_9.*

Proof By Theorem 1.6(b), such a G is conjugate to a subgroup of the standard inclusion $\mathrm{SO}_{4k} \subset \mathrm{SO}_{4k+1}$, acting transitively on S^{4k-1}. The only closed connected subgroups $G \subset \mathrm{SO}_{4k}$ acting transitively on S^{4k-1}, in the said dimensions, are the standard linear actions of SO_{4k}, U_{2k}, SU_{2k}, $\mathrm{Sp}_k\mathrm{Sp}_1$, $\mathrm{Sp}_k\mathrm{U}_1$ and Sp_k on $\mathbb{R}^{4k} = \mathbb{C}^{2k} = \mathbb{H}^k$, or the spin representation of Spin_7 on $\mathbb{C}^4$ (see eg [4, Examples 7.13, page 179]). But the groups $\mathrm{Sp}_k\mathrm{Sp}_1$, $\mathrm{Sp}_k\mathrm{U}_1$ and Sp_k for $k \geq 1$ cannot occur as structure groups of S^{4k+1}, since they contain the last one, Sp_k, which is excluded by Theorem 2.1 of [5]. □

Remark 3.8 For $n = 17$, the group $\mathrm{Spin}_9 \subset \mathrm{SO}_{16}$ acts transitively on S^{15}, but we do not know if the structure group of S^{17} could be reduced to it. For $n = 133$, as explained before, we do not know if the group $E_7 \subset \mathrm{SO}_{133}$ (or some subgroup of it acting irreducibly on $\mathbb{R}^{133}$) may appear as a reduction of the structure group of S^{133}.

3.4 The Dynkin index

Here we prove claim (b) from the proof of Lemma 3.6 of the previous subsection. We begin with some background.

Let $\rho\colon H \to G$ be a homomorphism of compact simple Lie groups. The third homotopy group of any simple Lie group is infinite cyclic (isomorphic to $\mathbb{Z}$), hence the induced map $\rho_*\colon \pi_3(H) \to \pi_3(G)$ has a cyclic cokernel of order $j \in \mathbb{N}$, called the *Dynkin index* of ρ (if $\rho_* = 0$ then $j = 0$, by definition). Clearly, j is *multiplicative*, ie if $\widetilde{H}$ is a simple compact Lie group and $\pi\colon \widetilde{H} \to H$ is a homomorphism, then $j(\rho\circ\pi) = j(\rho)j(\pi)$.

There is a simple Lie algebraic expression for $j(\rho)$. To state it, the Killing form on any simple compact Lie algebra needs to be normalized first by $\langle\delta,\delta\rangle = 2$, where δ is the longest root. Next, the pullback by $\rho\colon H \to G$ of the Killing form of G is an Ad_H–invariant quadratic form on the Lie algebra of H, hence, by simplicity of H, is a nonnegative multiple of the Killing form of H. This multiple turns out to be precisely the Dynkin index of ρ.

Theorem 3.9 *Let $\rho\colon H \to G$ be a homomorphism of compact simple Lie groups and $\rho_*\colon \mathfrak{h} \to \mathfrak{g}$ the induced Lie algebra homomorphism. Then*

$$\langle \rho_* X, \rho_* Y\rangle_{\mathfrak{g}} = j(\rho)\langle X, Y\rangle_{\mathfrak{h}} \tag{1}$$

for all $X, Y \in \mathfrak{h}$.

In fact, Dynkin *defined* $j(\rho)$ via formula (1) (see [8, formula (2.2), page 130]), and showed in the same article that $j(\rho)$ is an integer, without reference to its topological interpretation. Later, it was shown to have an equivalent definition via homotopy groups, as given above (we are not sure who proved it first; we learned it from [20, Section 2 of Chapter 5, page 257]).

Lemma 3.10 *$j(\rho) = 10$ for the irreducible representation $\rho\colon \mathrm{SO}_3 \to \mathrm{SO}_5$.*

Proof Theorem 3.9 gives an easy-to-follow recipe for j. To apply it, one needs to compute first the normalization of the Killing forms of SO_3 and SO_5.

Let $\mathfrak{so}_5$ be the set of 5×5 antisymmetric real matrices, the Lie algebra of SO_5, with $\mathfrak{t} \subset \mathfrak{so}_5$ the set of block diagonal matrices of the form $(x_1 J \oplus x_2 J \oplus 0)$, where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. The roots are $\pm x_1 \pm x_2$, $\pm x_1$ and $\pm x_2$, with $\delta := x_1 + x_2$. Since $\mathrm{tr}(XY)$ is clearly an Ad–invariant nontrivial bilinear form on $\mathfrak{so}_5$, the normalized Killing form of $\mathfrak{so}_5$ is of the form $\langle X, Y\rangle = \lambda\,\mathrm{tr}(XY)$ for some $\lambda \in \mathbb{R}$. The normalization condition is $\langle \delta^\flat, \delta^\flat\rangle = 2$, where $\delta^\flat \in \mathfrak{t}$ is defined via $\delta(X) = \langle\delta^\flat, X\rangle$ for all $X \in \mathfrak{t}$. Let $\delta^\flat = \lambda'(J \oplus J \oplus 0)$ for some $\lambda' \in \mathbb{R}$. Then, for all $X \in \mathfrak{t}$, $\langle\delta^\flat, X\rangle = \lambda\mathrm{tr}(\delta^\flat X) = -2\lambda\lambda'\delta(X)$, thus $-2\lambda\lambda' = 1$, so $\delta^\flat = -(1/2\lambda)(J\oplus J\oplus 0)$ and $2 = \langle\delta^\flat,\delta^\flat\rangle = \lambda\mathrm{tr}[(\delta^\flat)^2] = -1/\lambda$, hence $\lambda = -\frac{1}{2}$. It follows that $\langle X, Y\rangle_{\mathfrak{so}_5} = -\frac{1}{2}\,\mathrm{tr}(XY)$. For $\mathfrak{so}_3$ we get, by a similar argument, $\langle X, Y\rangle_{\mathfrak{so}_3} = -\frac{1}{4}\,\mathrm{tr}(XY)$.

Now let $\rho\colon \mathrm{SO}_3 \to \mathrm{SO}_5$ be the 5–dimensional irreducible representation on $\mathbb{R}^5$ (conjugation of traceless symmetric 3×3 matrices). Let $X = (J \oplus 0) \in \mathfrak{so}_3$. To calculate $\mathrm{tr}[(\rho_* X)^2]$, we let X act on $S^2((\mathbb{C}^3)^*)$ (complexifying, passing to the dual and adding an extra trivial summand does not affect trace). Now $x_1 \pm i x_2$ and x_3 are X–eigenvectors in $(\mathbb{C}^3)^*$, with eigenvalues $\pm i$ and 0, hence the eigenvalues of the $\rho_* X$–action on $S^2((\mathbb{C}^3)^*)$ are $\pm 2i$, $\pm i$, 0 and 0, and those of $(\rho_* X)^2$ are -4, -4, -1, -1, 0 and 0, giving $\mathrm{tr}[(\rho_* X)^2] = -10$. Thus $j(\rho) = \langle \rho_* X, \rho_* X\rangle_{\mathfrak{so}_5} / \langle X, X\rangle_{\mathfrak{so}_3} = 2\,\mathrm{tr}[(\rho_* X)^2]/\mathrm{tr}(X^2) = 10$, as claimed. □

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